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Exponential Decay of Resolvents and Discrete Eigenfunctions of Banded Infinite Matrices*

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In this paper, I prove exponential upper bounds on resolvents of banded infinite matrices acting on ℓ^p spaces, and on eigenfunctions of self-adjoint banded matrices corresponding to discrete eigenvalues. The method used is the theory of holomorphic families of type (A) in the sense of Kato.

1. INTRODUCTION

Let J be a centered m-banded matrix acting on $X=\ell^p(S)$, where $S=\mathbb{Z}$, $\mathbb{Z}^+$, or $\{1, ..., N\}$. Recall that for a centered m-banded matrix, m is even, and

$$a_{rs}=0 \qquad \text{if} \quad |r-s|>m/2$$

(note that a 2-banded matrix is also called tridiagonal). Let $T: X \to X$ be defined by

$$(Ty)(n)=y(n+1), \qquad \text{for} \quad y \in X$$

(note that if $S=\{1, ..., N\}$, then $(Ty)(n)=0$ for $n \geqslant N$). If $S=\mathbb{Z}$, then T is a bounded unitary operator. I shall let $V=T^*$. I may then write

$$J=B+\sum_{k=1}^{m/2} T^k A_k+\sum_{k=1}^{m/2} C_k V^k$$

where A_k, C_k, and B are diagonal matrices. There are two questions that I wish to address in this paper.

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(i) Given z in the resolvent set of J, what restrictions on the A_k, C_k, and B need to be made so that J satisfies a bound of the form

$$|(J-z)^{-1}(r,s)| \leqslant M_1 \exp(-M_2 |w(r)-w(s)|),$$

where M_1 and M_2 are positive numbers and w is a positive sequence (all of which depend on z)?

(ii) If λ is a discrete eigenvalue of a self-adjoint matrix J, under what restrictions on A_k and B will there be positive numbers C_1, μ, and a positive sequence w so that if y is an $\ell^2(S)$ eigenfunction, then

$$|y(n)| \leqslant C_1 \exp(-\mu w(n))$$

(where C_1, μ, and w depend on y and λ)?

I prove theorems in this paper which answer these questions.

Note that the answer to (ii) is well-known in the case of a second-order elliptic differential operator. The results which are the best-known are due to Agmon [1]. His results simplify and collect together earlier results, using essentially integration by parts and Hölder's inequality.

I also show that if J is a self-adjoint matrix with compact resolvent, and if the A_k satisfies certain conditions (see Theorem 3.3 below), then any eigenvector y in $\ell^2(S)$ corresponding to an eigenvalue λ has elements which decay faster then $\exp(-\mu w(n))$ for any $\mu > 0$. This leads to the following problem, which is related to (ii) above:

(iii) Find conditions on A_k and w, and a function f so that if λ is a discrete eigenvalue which is below the essential spectrum, and if $\mu < f(\text{dist}(\lambda, \sigma_{\text{ess}}(J)))$, then for an $\ell^2(S)$ eigenfunction y corresponding to λ, there is a constant $C_1 > 0$ such that

$$|y(n)| \leqslant C_1 \exp(-\mu w(n)).$$

Note that (iii) has been solved in the differential operator case in [1]; see the results in Simon [18–20] and Davies and Harrell [5] also.

The following theorem (see [8]) gives an answer to (i) for bounded J and $p=2$.

THEOREM [Demko, Moss, and Smith, 8]. *Let $J\colon X \to X$ be an m-banded bounded matrix, and suppose that $z \notin \sigma(J)$. Then there are constants $C > 0$, $0 < \gamma < 1$ such that*

$$|(J-z)^{-1}(r,s)| \leqslant C\gamma^{|r-s|}.$$

This result unified all the previous results which were proved mainly by workers in the theory of spline approximation (see [6, 7] and the references in [8]). The research which led to this article was inspired in part by this theorem. My results extend this theorem in two ways; first, to the case where $p \neq 2$, and second, to a large class of banded matrices which are possibly unbounded.

The method that I shall use to prove the main theorems below is due to Combes and Thomas [4] and is called the Combes–Thomas method in the literature of Schrödinger operators (see Reed and Simon [17]). This method was also used in [10, 21] to prove Theorem 3.1. I define $E(\kappa): X \to X$ by

$$(E(\kappa)\, y)(n) = \exp(i\kappa w(n))\; y(n),$$

which is a unitary operator for κ real. I then define $J(\kappa) = E(\kappa)\, JE(\kappa)^{-1}$, and find conditions on $w(n)$, A_k, and C_k under which $\{J(\kappa)\}$ forms a holomorphic family of type (A) in the sense of Kato (see [13, 17]). The proofs are very similar to the differential operator versions given in Simon [18–20].

In Section 2 I state some definitions and results which will be needed in the later sections, and in Section 3, prove the main results. In Section 4, I comment on the results proved here, and given some examples.

2. Holomorphic Families of Linear Operators

In this section, I give the definitions and theorems needed in the proofs to be given in Section 3. These results are taken primarily from [13], but some of them may also be found in [17].

Definition [Kato, 13]. Let X be a Banach space and $\{J(\kappa)\}$ a family of closed operators on X defined for κ in some neighborhood of 0. Then $\{J(\kappa)\}$ is a holomorphic family of linear operators of type (A) in a neighborhood on $\kappa = 0$ if the domain of $J(\kappa)$ is independent of κ, that is, $\mathscr{D}(J(\kappa)) = \mathscr{D}$, and if the power series

$$J(\kappa)\, y = Jy + \kappa J^{(1)}y + \kappa^2 J^{(2)}y + \cdots, \qquad y \in \mathscr{D}, \tag{2.1}$$

converges in the disk $|\kappa| < \rho$, where $\rho > 0$ is independent of y.

Note that this implies $J(\kappa)\, y$ is differentiable at $\kappa = 0$. If the family $\{J(\kappa)\}$ is a family of bounded operators, then this definition implies $\|J(\kappa)\|$ is differentiable (see [13]). The properties of such families of operators that I shall need are contained in the following theorem.

THEOREM 2.1 [13, 17, pp. 16, 22–23]. *Let $\{J(\kappa)\}$ be a family of linear operators depending on the complex parameter κ and let $J^{(p)}$, $p = 1, 2, \ldots$, be a sequence of closable linear operators. Suppose $\mathscr{D}(J^{(p)}) \supseteq \mathscr{D}(J)$, and that there are constants a, b, $c \geqslant 0$ with*

$$\|J^{(p)}y\| \leqslant c^{p-1}(a\,\|Jy\| + b\,\|y\|), \qquad p = 1, 2, \ldots .$$

(a) *The infinite series* (2.1) *converges for y in the domain of $J(\kappa)$ and $|\kappa| < (a+c)^{-1}$, and has a closed extension which forms a holomorphic family of type* (A) *for $|\kappa| < (a+c)^{-1}$.*

(b) *If $z \in \rho(J(0))$, then there is a number $\alpha > 0$ such that for $|\kappa| < \alpha$, $z \in \rho(J(\kappa))$, and the resolvent $(J(\kappa) - z)^{-1}$ is uniformly bounded in the disk $|\kappa| < \alpha$.*

(c) *If $J(\kappa)$ is self-adjoint for κ real and λ is a discrete eigenvalue of multiplicity p, then there is a $\gamma > 0$ and there are p not necessarily distinct single-valued functions $\lambda_1(\kappa), \ldots, \lambda_p(\kappa)$ holomorphic in $|\kappa| < \gamma$, with $\lambda_j(0) = \lambda$, so that $\lambda_1(\kappa), \ldots, \lambda_p(\kappa)$ are eigenvalues of $J(\kappa)$ for $|\kappa| < \gamma$. Furthermore, there are the only eigenvalues of $J(\kappa)$ for $|\kappa| < \gamma$.*

For part (a) of this theorem, see [17, p. 16], and for part (b) and (c), see [17, p. 22, 23]. Note that these results are also contained in Kato [13].

DEFINITION [Kato, 13]. Let F and G be linear operators on a Banach space $\mathscr{B}$. Then G is relatively bounded with respect to F, or G is F-bounded, if $\mathscr{D}(G) \supseteq \mathscr{D}(F)$, and if there are numbers a, $b \geqslant 0$ so that

$$\|Gy\| \leqslant a\,\|Fy\| + b\,\|y\|,$$

for $y \in \mathscr{D}(F)$. The smallest number $a \geqslant 0$ for which there is a b such that the above inequality holds is called the relative bound, or F-bound, of G with respect to F.

3. THE MAIN RESULTS

In [21], I proved the following theorem. It is a generalization of a result of [10].

THEOREM 3.1 [21]. *Let J be an m-banded matrix on X, suppose that $w(n)$ is a nonnegative sequence of real numbers so that the matrices*

$$W_k = \operatorname{diag}(w(n) - w(n+k)) \qquad \textit{and} \qquad U_k = \operatorname{diag}(w(n) - w(n-k))$$

are bounded, suppose that $W_k T^k A_k$ and $U_k C_k V^k$ are relatively bounded with respect to J, and suppose $z \notin \sigma(J)$. Then there are positive constants M_1 and M_2 such that

$$|(J-z)^{-1}(r, s)| \leqslant M_1 \exp(-M_2 |w(r)-w(s)|).$$

The proof of this result is not given, since I prove Theorem 3.2 below, which removes the hypothesis that the matrices W_k and U_k are bounded.

THEOREM 3.2 (compare to [10]). *Let J be an m-banded matrix on X, suppose that $w(n)$ is a nonnegative sequence of real numbers, and define the diagonal matrices*

$$W_k = \operatorname{diag}(w(n)-w(n+k)) \quad \text{and} \quad U_k = \operatorname{diag}(w(n)-w(n-k)).$$

Suppose that there are numbers a, b, $c \geqslant 0$ such that

$$\|(W_k)^p T^k A_k y\| \leqslant c^{p-1}(a \|Jy\| + b \|y\|),$$
$$\|(U_k)^p C_k V^k y\| \leqslant c^{p-1}(a \|Jy\| + b \|y\|),$$

$p = 1, 2, \ldots$, for y in the domain of J, and suppose $z \notin \sigma(J)$. Then there are positive constants M_1 and M_2 such that

$$|(J-z)^{-1}(r, s)| \leqslant M_1 \exp(-M_2 |w(r)-w(s)|).$$

Note that by taking $w(n) = n$, I recover the theorem of [10]. By taking the W_k and U_k to be bounded, the following inequalities show that Theorem 3.1 is a corollary of Theorem 3.2:

$$\|(W_k)^p T^k A_k y\| \leqslant \|W_k\|^{p-1} \|W_k T^k A_k y\|,$$
$$\|(U_k)^p C_k V^k y\| \leqslant \|U_k\|^{p-1} \|U_k C_k V^k y\|;$$

so we may take $c = \max\{\|W_k\|, \|U_k\|\}$ in Theorem 3.2. I show in the next section that Theorem 3.1 applies to a wider class of matrices J than the version of [10].

Proof. The proof uses the Combes–Thomas method as in [10], and as modified in [21]. I define $E(\kappa): X \to X$ by

$$(E(\kappa) y)(n) = \exp(i\kappa w(n))\, y(n)$$

(compare to [18, 19]). For any $y \in \mathscr{D}(J)$, let $J(\kappa): \mathscr{D}(J) \to X$ be defined by $(J(\kappa) y)(n) = (E(\kappa) J E(\kappa)^{-1} y)(n)$. Then

$$(J(\kappa) y)(n) = B + \sum_{k=1}^{m/2} e^{i\kappa(w(n)-w(n+k))} T^k A_k + \sum_{k=1}^{m/2} e^{i\kappa(w(n)-w(n-k))} C_k V^k,$$

or

$$
\begin{aligned}
(J(\kappa)\,y)(n) = (Jy)(n) + \sum_{k=1}^{m/2} \Bigl(& i\kappa(w(n)-w(n+k)) \\
& -\frac{\kappa^2}{2}(w(n)-w(n+k))^2 + \cdots \Bigr) T^k A_k\, y(n) \\
+ \sum_{k=1}^{m/2} \Bigl(& i\kappa(w(n)-w(n-k)) \\
& -\frac{\kappa^2}{2}(w(n)-w(n-k))^2 + \cdots \Bigr) C_k V^k y(n)
\end{aligned}
\tag{3.1}
$$

by expansion of the exponentials. By hypothesis, Theorem 2.1 (a) guarantees that $J(\kappa)$ has an extension to complex κ which forms a holomorphic family of type (A). Also, Theorem 2.1 (b) guarantees that there is $\beta > 0$ such that if $z \in \rho(J(0))$ then $z \in \rho(J(\kappa))$ for $|\kappa| < \beta$, and that the operator norm of $(J(\kappa)-z)^{-1}$ is uniformly bounded in κ for $|\kappa| < \beta$. A simple calculation shows that

$$(J(\kappa)-z)^{-1} = E(\kappa)(J-z)^{-1}\, E(\kappa)^{-1} \tag{3.2}$$

for κ real. Since $(J(\kappa)-z)^{-1}$ is a holomorphic family of type (A), then this identity must also be true for complex κ. By Theorem 2.1(b), there is a constant $M_1 > 0$ such that

$$|(J(\kappa)-z)^{-1}\,(r,s)| \leqslant \|(J(\kappa)-z)^{-1}\|_{\mathrm{op}} \leqslant M_1, \qquad |\kappa| < \beta, \tag{3.3}$$

since the operator norm dominates the absolute value of any element of a bounded matrix. Here, $\|\cdot\|_{\mathrm{op}}$ is the usual operator norm. Let $M_2 > 0$ be a number such that $|\kappa| = |iM_2| < \beta$. Then using this κ and (3.2) in (3.3) gives

$$|\exp(-w(r)\,M_2)(J-z)^{-1}\,(r,s)\exp(w(s)\,M_2)| \leqslant M_1,$$

and using κ^* and (3.2) in (3.3) gives

$$|\exp(w(r)\,M_2)(J-z)^{-1}\,(r,s)\exp(-w(s)\,M_2)| \leqslant M_1.$$

This proves the theorem. ∎

THEOREM 3.3 (compare to [18–20]). *Let $p=2$, let $J\colon X \to X$ be a self-adjoint m-banded matrix, let $w(n)$ be a nonnegative sequence such that the matrices*

$$W_k = \mathrm{diag}(w(n)-w(n+k)) \qquad \textit{and} \qquad U_k = \mathrm{diag}(w(n)-w(n-k))$$

are bounded, and suppose that $W_k T^k A_k$ and $U_k A_k V^k$ are relatively bounded with respect to J.

(a) *If J has compact resolvent, the relative bounds of $W_k T^k A_k$ and $U_k A_k V^k$ are zero, and $Jy = \lambda y$, $y \in X$ (so $\lambda \in \sigma_{\text{disc}}(J)$), then for any $\alpha > 0$ there is a constant $C_1 > 0$ such that*

$$|y(n)| \leqslant C_1 \exp(-\alpha w(n)).$$

(b) *If $\lambda \in \sigma_{\text{disc}}(J)$ and $Jy = \lambda y$ for $y \in X$, then for some constants μ, $C_2 > 0$,*

$$|y(n)| \leqslant C_2 \exp(-\mu w(n)). \tag{3.4}$$

Proof. I give a proof that is almost identical to the proof of a similar theorem for Schrödinger operators (see [17–20]). The key to this proof is the following lemma of O'Connor [15].

LEMMA [15 or 17]. *Let $E(\kappa): H \to H$ be a unitary group parametrized by $\mathbb{R}$. Suppose $P(\kappa)$ is a projection-valued holomorphic function which is given on a ball $|\kappa| < \beta$, $P(0)$ is of finite rank, and suppose for $|\eta| < \beta$, $|\kappa + \eta| < \beta$, η real*

$$E(\eta)\, P(\kappa)\, E(\eta)^{-1} = P(\kappa + \eta).$$

Then for any $y \in \operatorname{Range}(P(0))$, $E(\kappa)\, y$ has a holomorphic continuation from $\mathbb{R}$ to $\Delta = \{\kappa \in \mathbb{C} : |\operatorname{Im}(\kappa)| < \beta\}$.

For a proof of this, see [15] or [17, pp. 22, 23]. Note that $E(\kappa)\, y$ has a holomorphic continuation to Δ if and only if

$$\{\exp(aw(n))\, y(n)\} \text{ is in } X.$$

for any $a < \beta$.

As in the proof of Theorem 1, define $E(\kappa)$ and $J(\kappa)$. Then from (3.1) and (3.2), $J(\kappa)$ is a holomorphic family of type (A), for $|\kappa| < \rho$, where $\rho > 0$. Since λ is an eigenvalue of J of finite multiplicity, there are q eigenvalues $\lambda_1(\kappa), \ldots, \lambda_q(\kappa)$ in the discrete spectrum of $J(\kappa)$, and the $\lambda_j(\kappa)$ are analytic functions in $|\kappa| < \beta$, for some β possibly smaller than ρ. Now $J(\kappa)$ is unitarily equivalent to J for κ real, so $\lambda_j(\kappa) = \lambda$ for κ real, and thus $\lambda_j(\kappa) = \lambda$ for $|\kappa| < \beta$ (by analyticity). This type of argument is in [20], where it is used in the multiparticle Schrödinger case. Thus, $\lambda \in \sigma_{\text{disc}}(J(\kappa))$ for $|\kappa| < \beta$. Let

$$P(\kappa) = -(2\pi i)^{-1} \int_C R(J(\kappa), \omega)\, d\omega.$$

where $R(J(\kappa), \omega)$ is the resolvent of $J(\kappa)$, and C is the circle $|\omega-\lambda|=\varepsilon$ (note that ε must be chosen so that the circle contains only the eigenvalue λ and no other part of the spectrum of J). Then $P(0)$ is the projection onto the eigenspace associated with λ. Note that under the assumptions in either (a) or (b), $P(0)$ has finite rank.

To prove (b), note that for $|\eta|<\beta$, $|\eta+\kappa|<\beta$,

$$E(\eta)\, P(\kappa)\, E(\eta)^{-1}=P(\kappa+\eta).$$

By the lemma of O'Connor, for any $y \in \mathrm{Range}(P(0))$, $E(\kappa)\, y$ has a holomorphic continuation to Δ. By the statement immediately after the lemma, this proves that

$$\{\exp(aw(n))\, y(n)\} \text{ is in } X,$$

for any $a<\beta$.

To prove (a), note that if the resolvent of J is compact, then the resolvent of $J(\kappa)$, as defined in the proof of Theorem 3.1, is compact for κ real, and thus compact for all κ. Also, by assumption and Theorem 2.1(a), $J(\kappa)$ is an entire family of type (A), so $P(\kappa)$ has a holomorphic extension to Δ with $\beta=\infty$. From the lemma of O'Connor, this shows that for any $a>0$,

$$\{\exp(aw(n))\, y(n)\} \text{ is in } X,$$

so (a) is proved. ∎

Note that when $w(n)=n$, Theorem 3.3(b) is a discrete version of a theorem of Snol' (see [11]). There are also versions of these theorems which replace relative boundedness with relative form boundedness. This is connected with the definition of an operator J by a quadratic form, and requires $p=2$; that is, it is a Hilbert space theory. Also, it requires J to be m-sectorial; that is, the numerical range of J is contained in a sector of the complex plane. I shall not give these versions here, but let the reader formulate them. For the relevent theory of holomorphic families of quadratic forms, see [13 or 17].

For the differential operator $-\Delta+q$ on $L^2(\mathbb{R}^\nu)$, where q is bounded below, it is known that if $\mu>0$ satisfies

$$\mu^2<2(\inf \sigma_{\mathrm{ess}}(-\Delta+q)-\lambda),$$

where λ is a discrete eigenvalue below the essential spectrum, and if u is an $L^2(\mathbb{R}^\nu)$ eigenfunction of $-\Delta+q$ corresponding to λ, then there is a constant $C>0$ so that

$$|u(x)| \leqslant C \exp(-\mu r),$$

where $r = |x|$. This was first proven by Snol' (see [1, 11, and 20]). I conjecture that a similar statement is true for the infinite matrix case. Note that I have answered a special case of this in Theorem 3.2(b). In that case, any $\mu > 0$ will be a rate of decay. In [21], I began a study of this conjecture. and the results will appear in another paper.

4. Some Examples

In [8], Theorem 3.1 was proved with $w(n) = n$. I give some examples to show that Theorem 3.1 generalizes to cases not covered by the result of [8].

Example 1 (Laguerre polynomials [23]). Consider the difference equation

$$(n+1)\, y(n+1) - (2n+1)\, y(n) + ny(n-1) = -zy(n). \tag{4.1}$$

This is the difference equation for the Laguerre polynomials. Let $A_1 = \operatorname{diag}(n)$ and $B = \operatorname{diag}(2n+1)$. Then the matrix

$$J = B + TA_1 + T^*A_1$$

has spectrum $[0, \infty)$, since the weight function for the Laguerre polynomials is $\exp(-x)$. Suppose that $z \notin [0, \infty)$. It is known [14] that the resolvent matrix elements $(J-z)^{-1}(r, s)$ are given by

$$(J-z)^{-1}(r, s) = W^{-1} y_1(r_>)\, y_2(r_<), \tag{4.2}$$

where $r_> = \max\{r, s\}$, and $r_< = \min\{r, s\}$, and

$$W = y_1(n)\, y_2(n+1) - y_2(n)\, y_1(n+1)$$

is independent of n. It is also known that there are two solutions of (4.1), y^- and y^-, with

$$y^+(n; z) \sim n^{-1/4} \exp(2(n+1/2)^{1/2} \operatorname{Im}(-z)^{1/2})\, \phi_1(z),$$
$$y^-(n; z) \sim n^{-1/4} \exp(-2(n+1/2)^{1/2} \operatorname{Im}(-z)^{1/2})\, \phi_2(z),$$

where ϕ_1 and ϕ_2 are functions analytic on the cut plane $\mathbb{C} \setminus [0, \infty)$. This is because solutions of (4.1) can be written as confluent hypergeometric functions (see Masson [14]). Using these results in (4.2) I obtain

$$|(J-z)^{-1}(r, s)| \leqslant Cr^{-1/4} s^{-1/4} \exp(-\operatorname{Im}(-z)^{1/2} |r^{1/2} - s^{1/2}|),$$

for r and s large enough, where $C>0$ is a constant which depends on z. In fact, the asymptotic results quoted show that if I divide the left-hend side by the right-hand side and take the limit as r and s tend to ∞, I get 1. Thus, this is the best possible bound. To show that Theorem 3.1 applies in this case, I use the following argument to choose the sequence $w(n)$. Note that

$$TA_1\, y(n) = (n+1)\, y(n+1) \qquad \text{and} \qquad A_1 T^* y(n) = ny(n-1).$$

I shall choose $w(n)$ so that $W_1 TA_1$ and $U_1 A_1 T^*$ are bounded. To do this requires that I solve

$$w(n+1) - w(n) = n^{-1}, \qquad n = 1, 2, \ldots .$$

A solution of this is the digamma function $\psi(n)$ defined by

$$\psi(n) = \Gamma'(n)/\Gamma(n).$$

From [9],

$$\psi(n) = \log(n-1) - (2n)^{-1} + o(1), \qquad \text{as} \quad n \to \infty.$$

Thus, I shall take $w(n) = \psi(n)$. Then it is immediate that $W_1 TA_1$ and $U_1 A_1 T^*$ are bounded matrices. Thus, they are relatively bounded with respect to J. Theorem 3.1 applies and shows that there are positive constants M_1 and M_2 such that

$$|(J-z)^{-1}(r, s)| \leqslant M_1 \exp(-M_2 |\psi(r) - \psi(s)|).$$

This bound is not as good as the best bound obtained above. This result also shows that Theorem 3.1 with $w(n) = 1$ (that is, the result of [8]) does not apply to this matrix J. Indeed, if Theorem 3.1 holds with $w(n) = 1$, then it gives the bound

$$|(J-z)^{-1}(r, s)| \leqslant M_1 \exp(-M_2 |r-s|).$$

But this is better than the best bound given above, and yields a contradiction. Thus, Theorem 3.1 applies when the result of [8] (that is, Theorem 3.1 with $w(n) = n$) does not. A referee has suggested the following alternate proof of this result, which I only sketch here. If TA_1 and $A_1 T^*$ were relatively bounded with respect to J, then for sufficiently small $\varepsilon > 0$, $J + \varepsilon(TA_1 + A_1 T^*)$ would be bounded below. But this contradicts the fact that, for the Meixner–Pollaczek polynomials, the spectrum is $(-\infty, \infty)$. Note that the conclusion of Theorem 3.1 for the Laguerre polynomials is not the best possible.

The next example shows that Theorem 3.1 can give the best possible bound on the resolvent.

EXAMPLE 2 (Hermite polynomials [23]). Consider the difference equation

$$(n+1)^{1/2}\, y(n+1) + n^{1/2} y(n-1) = zy(n). \tag{4.3}$$

This is the difference equation for the Hermite polynomials. Let $A_1 = \operatorname{diag}(n^{1/2})$ and $B = 0$. Then the matrix

$$J = TA_1 + A_1 T^*$$

has spectrum $(-\infty, \infty)$, since the Hermite polynomials are orthogonal on $(-\infty, \infty)$ with respect to the weight function $\exp(-x^2)$. Suppose $z \notin (-\infty, \infty)$. Equation (4.3) can be solved in terms of the parabolic cylinder functions (see Masson [14]), and by using (4.2), the best possible bound on $(J-z)^{-1}(r, s)$ is given by

$$|(J-z)^{-1}(r,s)| \leqslant Cr^{-1/4}s^{-1/4} \exp(-\operatorname{Im}(z)\, |r^{1/2} - s^{1/2}|)\, \phi(z),$$

for $\operatorname{Im}(z) > 0$, and

$$|(J-z)^{-1}(r,s)| \leqslant Cr^{-1/4}s^{-1/4} \exp(\operatorname{Im}(z)\, |r^{1/2} - s^{1/2}|)\, \psi(z),$$

for $\operatorname{Im}(z) < 0$, where ψ and ϕ are functions analytic for $\operatorname{Im}(z) \neq 0$, and the bounds hold for all r, s large enough. Here, $C > 0$ is a constant. That this is the best possible bound is shown by the asymptotic results found in Erdélyi [9], and by an argument similar to that used in Example 1. Let $w(n) = n^{1/2}$. Then it easily seen that $W_1 TA_1$ and $U_1 A_1 T^*$ are bounded, so they are relatively bounded with respect to J. Thus, Theorem 3.1 applies, and there are positive constants M_1 and M_2 such that

$$|(J-z)^{-1}(r,s)| \leqslant M_1 \exp(-M_2\, |r^{1/2} - s^{1/2}|).$$

In a manner similar to that of Example 1, it can be shown that the result of [8] does not apply to this example.

EXAMPLE 3 (Charlier polynomials [14, 22]). Now consider the difference equation

$$(n+1)^{1/2}\, y(n+1) - (z-n-1)\, y(n) + n^{1/2} y(n-1) = 0. \tag{4.4}$$

This is the difference equation associated with the Charlier polynomials. Let A_1 be the same as in Example 2, and let $B = \operatorname{diag}(n-1)$. Then the matrix

$$J = B + TA_1 + A_1 T^*$$

has purely discrete spectrum $\{0, 1, 2, ...\}$. This follows since the Charlier polynomials are orthogonal on this discrete set with respect to the weight $(n!)^{-1}$ (see [14]). Note that these points in the spectrum are discrete eigenvalues with multiplicity at most 1. Using $w(n) = n^{1/2}$ again shows that Theorem 3.3(b) applies. Thus, if λ is in the spectrum of J, and $Jy = \lambda y$, where y is in $\ell^2(\mathbb{Z}^+)$, then there are constants C, $\mu > 0$ such that

$$|y(n)| \leqslant C \exp(-\mu n^{1/2}).$$

This result appears to be new.

EXAMPLE 4 (attractive Coulomb potential polynomials [2]). Consider the radial Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{Z}{r},$$

where Z is the electronic charge. I assume that $Z < 0$, so the potential is attractive. Note that this is the physical range of values of Z. Let ψ be a solution of $(H - E)\psi = 0$. In [2], ψ is expanded in the series

$$\psi(r) = \sum_{n=0}^{\infty} (n+2)^{-1} b_n(x)\, \phi_n(r),$$

where

$$a = 2Z,$$
$$x = (E - 1/8)(E + 1/8)^{-1},$$
$$\phi_n(r) = re^{-r/2} L_n^1(r)$$

(L_n^α are the generalized Laguerre polynomials), and the $b_n(x)$ satisfy the difference equation

$$(n+1)\, b_{n+1}(x) - 2[(n+a)x - a]\, b_n(x) + nb_{n-1}(x) = 0.$$

This difference equation is a special case of the difference equation satisfied by the Pollaczek polynomials,

$$b_n(x) = P_n^1(x; 2Z, -2Z),$$

where $P_n^\lambda(x; a, b)$ are the Pollaczek polynomials (see [2, 22]). These polynomials are orthogonal on $(-\infty, \infty)$, and the explicit orthogonality is given in Section 4 of [2]. Note that I am assuming that a is in Region II in the language of [2], that is,

$$-1/2 < a < 0,$$

since this is the physical value of the parameter. Let

$$J = B + TA + AT^*$$

be the infinite matrix with A and B the diagonal matrices defined by

$$A = \operatorname{diag}\left(\frac{n}{2(n+a)^{1/2}(n+a-1)^{1/2}}\right),$$

$$B = \operatorname{diag}\left(\frac{a}{n+a}\right).$$

The spectrum of J consists of the interval $[-1, 1]$ and discrete eigenvalues $x_p < -1$ given by

$$x_p = [a^2 + (p+1/2)^2]/[a^2 - (p+1/2)^2],$$

and $x_p \to -1$ as $p \to \infty$ (see [2]). Let $w(n) = n$. Then Theorem 3.3(b) applies, so if $y \in \ell^2(\mathbb{Z}^+)$ solution of $Jy = x_p y$, then there are constants C, $\mu > 0$ so that

$$|y(n)| \leqslant C \exp(-\mu n).$$

This result was not given in [2].

There are other examples of Theorems 3.1 and 3.3 that I could give here. For instance, consider the radial part of the quantum mechanical harmonic oscillator, whose Hamiltonian H is

$$H = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{2r^2} + (C + 1/2)\, r^2.$$

As shown in [2], this Hamiltonian can be diagonalized by using the properties of the generalized Laguerre polynomials. The resulting difference equation is a special case of the Meixner, or Meixner–Pollaczek polynomials, depending on the sign of $1 + 1/C$. For $C > -1/2$, that is for the physical case, the corresponding polynomials are the Meixner polynomials, and the spectrum is discrete, as is well-known (see [2] and the references there for these results). Other examples from the literature are the polynomials studied by Carlitz [3], the generalized Chebyshev polynomials studied by Ismail and Mulla [12], the Heun and Hautot polynomials [24], and the Wilson polynomials [25]. It has been shown by Tater [23] that the Hamiltonian

$$H = -\frac{d^2}{dx^2} + ax^2 + bx^4 + cx^6, \qquad c > 0,$$

can be associated with a three-term recurrence relation, that is, a difference equation. Note that an application of Wilson's polynomials to quantum mechanical scattering has been given in [16].

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